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INSTANTONS AND THE GEOMETRY OF THE NILPOTENT VARIETY

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1. Introduction

Let $\mathfrak{g}$ be the Lie algebra of a compact, connected, semisimple Lie group G, and let $\varphi\colon \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}\to\mathbf{R}$ be the function

$$\varphi(A_1, A_2, A_3) = \sum_1^3 \langle A_i, A_i\rangle + \langle A_1, [A_2, A_3]\rangle,$$

where $\langle\ ,\ \rangle$ is an Ad-invariant inner product. We are going to study the trajectories of the gradient flow of φ. It turns out that the space of bounded trajectories is closely related to the *nilpotent variety*:

$$\mathcal{N} = \{x \in \mathfrak{g}^c \mid \mathrm{ad}(x) \text{ is nilpotent}\}.$$

Here $\mathfrak{g}^c$ is the complex Lie algebra $\mathfrak{g}\otimes\mathbf{C}$. On the other hand φ exhibits symmetries which are not immediately visible in $\mathcal{N}$; for example, the obvious action of SO(3) on $\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}$ leaves φ invariant. Exploiting this we obtain some new information (and some old information) about the geometry of the nilpotent variety, enough to give an explanation for Brieskorn's result [1] that $\mathcal{N}$ has a finite quotient singularity along the codimension-2 orbits. We discuss Brieskorn's result and its relationship to the SO(3) action in §3. As a spin-off we find that the icosahedral group (for example) occurs naturally as the intersection of two three-dimensional subgroups, copies of SO(3), inside the compact group of type E_8.

The results concerning the trajectories of φ are stated in §2 and proved in §§4–6. Some standard information about nilpotent elements in semisimple Lie algebras is summarized in an appendix.

2. The gradient flow

To motivate the results of this section, it will be helpful to give a geometric interpretation of φ. We identify $\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}$ with the space of linear

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maps $L(\mathfrak{su}(2), \mathfrak{g})$ by assigning to (A_1, A_2, A_3) the map $A: e_i \mapsto A_i$. Here (e_1, e_2, e_3) is a basis for $\mathfrak{su}(2)$ satisfying the relations

$$-2e_1 = [e_2, e_3], \qquad -2e_2 = [e_3, e_1], \qquad -2e_3 = [e_1, e_2].$$

For definiteness, we shall take

$$e_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

If we think of $\mathfrak{su}(2)$ as the space of left-invariant vector fields, then A becomes a Lie algebra-valued one-form, or connection matrix, on the three-manifold $\mathrm{SU}(2)$. Being left-invariant, it is naturally an invariant connection in a principal G-bundle $P \to \mathrm{SU}(2)$ with $\mathrm{SU}(2)$ action. The number $\varphi(A)$ can now be interpreted as the Chern-Simons invariant of this connection, calculated using a left-invariant trivialization of the bundle. It is a simple observation, successfully exploited in [4], that in a temporal gauge, the anti-self-dual Yang-Mills equations on a cylinder $Y^3 \times \mathbf{R}$ coincide with the gradient flow equations for the Chern-Simons functional on the three-manifold Y. (A *temporal gauge* for a connection A on $Y \times \mathbf{R}$ is one in which the component in the $\mathbf{R}$ direction $\langle A, \partial/\partial t\rangle$ is zero.) In our situation this means that the trajectories of $-\nabla\varphi$ can be interpreted as anti-self-dual connections on the four-manifold $\mathrm{SU}(2) \times \mathbf{R}$ which are invariant under an action of $\mathrm{SU}(2)$.

Written fully, the equation $\dot{A} = -\nabla\varphi(A)$ is:

(1)
$$\dot{A}_1 = -2A_1 - [A_2, A_3], \quad \dot{A}_2 = -2A_2 - [A_3, A_1], \quad \dot{A}_3 = -2A_3 - [A_1, A_2].$$

The following observations are more-or-less immediate:

(i) the critical points of φ are the Lie algebra homomorphisms ρ: $\mathfrak{su}(2) \to \mathfrak{g}$; i.e., triples satisfying $-2A_1 = [A_2, A_3]$, etc.;

(ii) the zero homomorphism is a local minimum of φ;

(iii) at all critical points other than 0, the value of φ is positive.

(The value of φ at a homomorphism ρ is, up to an overall factor, the *index* of ρ as defined by Dynkin (see [3], in which can be found a table listing all homomorphisms $\rho: \mathfrak{su}(2) \to \mathfrak{g}$, together with their indices, for all the exceptional Lie algebras).)

For each Lie algebra homomorphism $\rho: \mathfrak{su}(2) \to \mathfrak{g}$, let $C(\rho) \subset L(\mathfrak{su}(2), \mathfrak{g})$ denote the critical manifold consisting of all homomorphisms which are conjugate to ρ under the adjoint action of G. For each pair of homomorphisms ρ_-, ρ_+, let $M(\rho_-, \rho_+)$ denote the space of solutions $A(t)$ to the gradient flow equations (1) satisfying the following boundary conditions:

(2) $$\lim_{t\to-\infty} A(t) \in C(\rho_-), \qquad \lim_{t\to+\infty} A(t) = \rho_+.$$

Note that we only specify the conjugacy class of the limit in the backward direction, but specify the limit itself in the forward direction. Note also that we consider parametrized trajectories, so there is an action of $\mathbf{R}$ on $M(\rho_-, \rho_+)$ which replaces $A(t)$ by $A(t+\lambda)$.

Again, we can motivate this definition by looking at the four-dimensional interpretation. The boundary conditions (2) imply that the corresponding anti-self-dual connection over $\mathrm{SU}(2)\times\mathbf{R}$ extends to the conformal compactification S^4 after a gauge transformation. So we are now studying invariant 'instanton' connections in a G-bundle $P \to S^4$ with SU(2) action. Over the fixed points 0 and ∞ in S^4, the group SU(2) acts on the fibers of P, thus giving two homomorphisms $\mathrm{SU}(2)\to G$. It is not difficult to see that these group homomorphisms are obtained from ρ_- and ρ_+ by exponentiation. Our space $M(\rho_-, \rho_+)$ is not quite the instanton moduli space in this situation: there remains an action of the centralizer of ρ_+, and the instanton moduli space is the quotient of $M(\rho_-, \rho_+)$ by this action. Another way to say this is that $M(\rho_-, \rho_+)$ can be identified with the 'framed' instanton moduli space, consisting of invariant anti-self-dual connections in P modulo the SU(2)-invariant gauge transformations which are 1 at infinity.

Our main result provides a description of the spaces $M(\rho_-, \rho_+)$. First we need some preliminary definitions. Let $\rho: \mathfrak{su}(2)\to\mathfrak{g}$ be a Lie algebra homomorphism and write

$$H = \rho\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad X = \rho\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \qquad Y = \rho\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

where we have extended ρ to a homomorphism of complex algebras, $\rho: \mathfrak{sl}(2,\mathbf{C})\to\mathfrak{g}^c$. We define $\mathcal{N}(\rho)\subset\mathfrak{g}^c$ as the family of all nilpotent elements in $\mathfrak{g}^c$ which are conjugate to Y under the adjoint action of G^c. This is a smooth submanifold of $\mathfrak{g}^c$. We define $S(\rho)\subset\mathfrak{g}^c$ to be the affine subspace

(3) $$S(\rho) = Y + z(X),$$

where $z(X)$ denotes the centralizer of X in $\mathfrak{g}^c$.

Remark. The definition of $S(\rho)$ appeared in [9]. It is a transverse slice to the orbit $\mathcal{N}(\rho)$ at the point Y, as one can see by applying the representation theory of $\mathfrak{sl}(2,\mathbf{C})$ to the action of $\langle H, X, Y\rangle$ on $\mathfrak{g}^c$. Moreover, it has the following global properties:

(i) $S(\rho)$ meets $\mathcal{N}(\rho)$ only at Y;

(ii) $S(\rho)$ meets only those nilpotent orbits whose closures contain $\mathcal{N}(\rho)$, and its intersection with those orbits is transverse.

These properties are consequences of the fact that there is an action of the scalars on $\mathfrak{g}^c$ which preserves both $\mathcal{N}(\rho)$ and $S(\rho)$, and which has positive weights on the latter. So the behavior of the intersections globally is determined by the behavior near Y. Proofs of these assertions are in [9].

Now we can state the main theorem.

Theorem 1. *For any pair of homomorphisms ρ_-, ρ_+, there is a natural diffeomorphism*

$$M(\rho_-, \rho_+) \cong \mathcal{N}(\rho_-) \cap S(\rho_+).$$

Remarks. (i) When ρ_+ is the trivial representation, $S(\rho_+)$ is all of $\mathfrak{g}^c$. So in this case $M(\rho_-, \rho_+)$ is identified with the adjoint orbit $\mathcal{N}(\rho_-)$.

(ii) The word 'natural' in the statement of the theorem hides the fact that an important choice must be made: the diffeomorphism depends on the choice of a direction in $\mathbf{R}^3$, or equivalently an oriented decomposition $\mathbf{R}^3 = \mathbf{R} \oplus \mathbf{C}$. Since $\mathcal{N}(\rho_-) \cap S(\rho_+)$ is a complex manifold, $M(\rho_-, \rho_+)$ therefore obtains a family of complex structures parametrized by the two-sphere. In fact, $M(\rho_-, \rho_+)$ has a natural hyper-Kähler structure: this is a reflection of the rather general hyper-Kähler property of moduli spaces associated with the anti-self-dual Yang-Mills equations (see [5]).

(iii) Every orbit of nilpotent elements occurs as $\mathcal{N}(\rho)$ for some homomorphism $\rho: \mathfrak{su}(2) \to \mathfrak{g}$, and the assignment of $\mathcal{N}(\rho)$ to ρ establishes a one-to-one correspondence between the nilpotent orbits and the conjugacy classes of homomorphisms ρ. This is explained in the Appendix as Proposition A3.

3. Brieskorn's theorem

We recall that the nilpotent variety $\mathcal{N}$ consists of finitely many orbits of G^c, and that there is a unique orbit, the *regular orbit*, which is open and dense in $\mathcal{N}$: its complex dimension is $\dim G^c - \operatorname{rank} G^c$. This fact is due to Kostant [6], who originally used the term *principal* orbit. When G is simple, there is a unique nilpotent orbit of dimension $\dim G^c - \operatorname{rank} G^c - 2$. This is the *sub-regular* orbit [11]; it has complex codimension 2 in $\mathcal{N}$. By Proposition A3, there exist homomorphisms $\rho, \rho': \mathfrak{su}(2) \to \mathfrak{g}$ such that $\rho\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $\rho'\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ lie in the regular and sub-regular orbits respectively. Let $S(\rho')$ be the transverse slice to the sub-regular orbit defined by (3). The following result was part of a conjecture made by Grothendieck. A proof was given by Brieskorn [1] and a fuller treatment by Slodowy [9].

Proposition 2(a). *The transverse slice* $\mathcal{N} \cap S(\rho')$ *is bihomomorphic to* $\mathbf{C}^2/\Gamma$, *where* Γ *is a finite subgroup of* $SU(2)$.

Thus $\mathcal{N}$ has a finite quotient singularity along the sub-regular orbit. The statement of the full result goes on to determine which finite groups Γ occur. For simple Lie groups of type A, D, E, one obtains the following list, which provides the well-known correspondence between the finite subgroups of $SU(2)$ and the simply-laced Dynkin diagrams:

(4)

G	Γ
A_r	cyclic of order $r+1$
D_r	binary dihedral of order $4r-8$
E_6	binary tetrahedral
E_7	binary octahedral
E_8	binary icosahedral

In view of Theorem 1, we now have a differential-geometric interpretation of Proposition 2(a):

Proposition 2(b). *Let* ρ *and* ρ' *be homomorphisms* $\mathfrak{su}(2) \to \mathfrak{g}$ *associated with the regular and sub-regular nilpotent orbits. Then the space of trajectories* $M(\rho, \rho')$ *is diffeomorphic to* $(S^3/\Gamma) \times \mathbf{R}$, *where* $\Gamma \subset SU(2)$ *is a finite group.*

So far, this is just a restatement of part of Brieskorn's theorem in terms of the trajectories of φ. We have even lost much of the strength of the original, for we make no mention of the complex analytic structure. However, while Proposition 2(a) may seem a little mysterious, Proposition 2(b) in contrast is entirely unsurprising, as we now show.

Both $SU(2)$ and G act on $L(\mathfrak{su}(2), \mathfrak{g})$ by their adjoint representations. While the second action is also clearly visible in the nilpotent variety, the first is not. Let $\rho, \rho' : \mathfrak{su}(2) \to \mathfrak{g}$ be as above, and let $R, R' : SU(2) \to G$ be obtained from these by exponentiation. Consider now the action of $SU(2)$ on $L(\mathfrak{su}(2), \mathfrak{g})$ defined by

$$A \mapsto \mathrm{Ad}(R'(u)) \circ A \circ \mathrm{Ad}(u^{-1}), \tag{5}$$

for $u \in SU(2)$. This action preserves φ and fixes ρ'. It therefore gives an action of SU(2) on the trajectory space $M(\rho, \rho')$. Thus $M(\rho, \rho')$ is a manifold of dimension four (by Theorem 1), admitting a free action of $\mathbf{R}$ (by reparametrizing the trajectories), as well as a commuting action of SU(2). Proposition 2(b) is immediate if we allow that $M(\rho, \rho')$ is connected and that SU(2) acts with three-dimensional orbits. We cannot prove connectedness directly, but we can easily see that the orbits are three-dimensional. Indeed, we can go some way towards describing the stabilizers for the SU(2) action, as follows.

We introduce a strict partial order on the set of homomorphisms ρ by declaring that $\rho > \rho'$ if there exists a nonconstant solution $A(t)$ for the equations (1) with

$$A(t) \to \rho \quad \text{as } t \to -\infty, \qquad A(t) \to \rho' \quad \text{as } t \to +\infty.$$

(Note that we mean ρ and ρ' to refer to the actual homomorphisms rather than their conjugacy classes.) Let ρ' be a homomorphism associated with the sub-regular orbit, and define $\Lambda = \{\rho | \rho > \rho'\}$. All homomorphisms $\rho \in \Lambda$ are associated with the regular orbit, for that is the only nilpotent orbit whose closure strictly contains the sub-regular orbit. Formula (5) defines an action of SU(2) on Λ:

$$\rho \mapsto \mathrm{Ad}(R'(u)) \circ \rho \circ \mathrm{Ad}(u^{-1}).$$

Since ρ is a Lie algebra homomorphism, this can also be written as

$$\rho \mapsto \mathrm{Ad}(R'(u)) \circ \mathrm{Ad}(R(u^{-1})) \circ \rho, \tag{6}$$

where R is obtained from ρ by exponentiation. It is a basic property of any representation ρ associated with the regular orbit that its centralizer in the (compact) adjoint group $\mathrm{Ad}(G)$ is trivial (see [9, p. 116]). We immediately have:

Lemma 3. *In the action of* SU(2) *on* Λ *given by* (6), *the stabilizer of* $\rho \in \Lambda$ *is the subgroup*

$$\hat{\Gamma} = \{u \in \mathrm{SU}(2) | \mathrm{Ad}(R(u)) = \mathrm{Ad}(R'(u))\}.$$

Since ρ and ρ' are not conjugate in $\mathfrak{g}$, their images can intersect only at zero (a consequence of Lemma A2). It therefore follows that the stabilizer described in Lemma 3 is finite. Now, the stabilizer of any trajectory is certainly contained in the stabilizer of its endpoint. So we deduce that the stabilizers for the SU(2) action on $M(\rho, \rho')$ are finite also; indeed, they are conjugates of a subgroup of the group $\hat{\Gamma}$.

Of course, the stabilizers Γ for the action of SU(2) on $M(\rho, \rho')$ are identified for us in the table (4). In the case of E_7 or E_8, the group Γ

is a maximal finite subgroup of SU(2), so in these cases $\hat{\Gamma}$ must coincide with Γ. Spelling this out in the case of E_8, for example, we have:

Theorem 4. *Let G be the compact Lie group of type E_8, let $\rho, \rho' : \mathfrak{su}(2) \to \mathfrak{g}$ be homomorphisms associated with the regular and sub-regular nilpotent orbits in $\mathfrak{g}^c$, and let these be positioned within their conjugacy classes so that $\rho > \rho'$ in the above sense. Let R, R' be the corresponding homomorphisms $\mathrm{SU}(2) \to G$. Then the group*

$$\Gamma = \{u \in \mathrm{SU}(2) | R(u) = R'(u)\}$$

is the binary icosahedral group.

Remarks. (i) The images of R and R' are copies of SO(3) in G. Their intersection is the symmetry group of the icosahedron.

(ii) The equations (1) enter the statement of the theorem only through the definition of the partial order. It would be interesting to have an algebraic characterization of the relation $\rho > \rho'$, and then to obtain a verification of Theorem 4 which was independent of the analytic result Theorem 1.

(iii) Of course, one must conjecture that Theorem 4 extends to all the types in the A, D, E classification. (G should be the adjoint group.) As we have mentioned, the E_7 case yields to the same argument, because the binary octahedral group is maximal. In the case of E_6, Slodowy [10] has pointed out how one can rule out the possibility that the group $\hat{\Gamma}$ defined in Lemma 3 is strictly larger than the binary tetrahedral group; so Theorem 4 extends to this case also.

(iv) The arguments used here can be applied to other adjoint orbits. First of all, it is not hard to show that SU(2) always acts on $M(\rho_-, \rho_+)$ with finite stabilizers, except in the trivial case where $\rho_+ \in C(\rho_-)$. So, as long as $M(\rho_-, \rho_+)$ is four-dimensional, each component of $M(\rho_-, \rho_+)$ is diffeomorphic to $(S^3/\Gamma) \times \mathbf{R}$ for some $\Gamma \subset \mathrm{SU}(2)$. In terms of the nilpotent orbits, this means that if $\mathscr{O}$ is an orbit whose closure contains an orbit $\mathscr{O}'$ with complex codimension two, then the link of $\mathscr{O}'$ in the closure of $\mathscr{O}$ is a disjoint union of homogeneous spaces, $(S^3/\Gamma_1) \cup \cdots \cup (S^3/\Gamma_p)$. In the case of the classical groups, this observation forms a small part of the results of [7].

In conclusion, let us consider again the hyper-Kähler structure on $M(\rho_-, \rho_+)$ which was mentioned in the second remark following the statement of Theorem 1. The action of SU(2) which we have been considering induces a nontrivial action on the two-sphere family of complex structures, preserving none of them. This is why the SU(2) action is difficult to describe in terms of the nilpotent variety: the diffeomorphism in Theorem 1 involves a choice of complex structure, and this breaks the SU(2) symmetry.

In a later paper, the author hopes to extend Theorem 1. By considering not just the bounded trajectories of the gradient flow, but also some trajectories which grow exponentially as $t \to -\infty$, one can show that all adjoint orbits in $\mathfrak{g}^c$, not just the nilpotent ones, arise as moduli spaces. A consequence is that all adjoint orbits have hyper-Kähler structures. Along these lines, the results of this paper can be related to the constructions in [8].

4. Analysing the ODE: complex trajectories

If we make the substitution $T_i = e^{2t} A_i$ and set $s = -\frac{1}{2} e^{-2t}$, then the gradient-flow equations (1) become

$$\frac{dT_1}{ds} = -[T_2, T_3], \text{ etc.}$$

These are *Nahm's equations*, and were studied in [2]. Our proof of Theorem 1 follows [2] very closely; we are studying the same equations, but with slightly different boundary conditions. To begin, we extend the equations, introducing a fourth Lie algebra-valued function $A_0(t)$ and writing

$$\begin{aligned} \dot{A}_1 &= -2A_1 - [A_0, A_1] - [A_2, A_3], \\ \dot{A}_2 &= -2A_2 - [A_0, A_2] - [A_3, A_1], \\ \dot{A}_3 &= -2A_3 - [A_0, A_3] - [A_1, A_2]. \end{aligned} \tag{7}$$

These new equations are invariant under an action of the group $\mathscr{G}$ of all smooth maps $g: \mathbf{R} \to G$, given by

$$\begin{aligned} A_0 &\mapsto \mathrm{Ad}(g)(A_0) - \frac{dg}{dt} \cdot g^{-1}, \\ A_i &\mapsto \mathrm{Ad}(g)(A_i), \qquad i = 1, 2, 3. \end{aligned} \tag{8}$$

Such a 'real gauge transformation' g can always be chosen so as to make $A_0 \equiv 0$, in which case (7) reduce to the previous equations (1); so the problem is not essentially changed. (The boundary conditions do not yet concern us.) Next we break the natural symmetry of the problem by introducing complex coordinates and writing

$$\alpha = \tfrac{1}{2}(A_0 + iA_1), \qquad \beta = \tfrac{1}{2}(A_2 + iA_3). \tag{9}$$

So α and β are now functions with values in $\mathfrak{g}^c$. The three equations

(7) can be written in terms of α and β as one 'real' equation and one 'complex' equation:

(10)(a)

$$\hat{F}(\alpha, \beta) := \frac{d}{dt}(\alpha + \alpha^*) + 2(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0,$$

(10)(b)

$$\frac{d\beta}{dt} + 2\beta + 2[\alpha, \beta] = 0.$$

The complex equation (10)(b) is invariant under a larger group, the 'complex gauge group' $\mathscr{G}^c$ of smooth maps $g\colon \mathbf{R} \to G^c$. We write $(\alpha', \beta') = g(\alpha, \beta)$, and the action is determined by

$$\alpha' = \mathrm{Ad}(g)(\alpha) - \frac{1}{2}\frac{dg}{dt}g^{-1}, \qquad \beta' = \mathrm{Ad}(g)(\beta).$$

Let $\rho_-, \rho_+\colon \mathfrak{su}(2) \to \mathfrak{g}$ be two Lie algebra homomorphisms, and write

$$H_- = \rho_-\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_- = \rho_-\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y_- = \rho_-\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with similar notation for H_+, X_+, and Y_+. Mimicking (1.21) and (1.22) from [2], we make the following definition.

Definition 5. A *complex trajectory* associated with the homomorphisms ρ_-, ρ_+ is a pair of smooth functions $\alpha, \beta\colon \mathbf{R} \to \mathfrak{g}^c$ such that:

(i) the complex equation (10)(b) holds,

(ii) in the limit as $t \to +\infty$,

$$2\alpha(t) \to H_+, \qquad \beta(t) \to Y_+,$$

(iii) in the limit as $t \to -\infty$,

$$2\alpha(t) \to \mathrm{Ad}(g)(H_-), \qquad \beta(t) \to \mathrm{Ad}(g)(Y_-)$$

for some $g \in G$, the compact group,

(iv) α and β approach their limits at $t = \pm\infty$ with exponential decay; i.e., $|2\alpha(t) - H_+| < Ke^{-\eta t}$ for some $\eta > 0$, etc.

Definition 6. Two complex trajectories (α, β) and (α', β') are *equivalent* if there exists $g\colon \mathbf{R} \to G^c$, a bounded path with $g(t) \to 1$ as $t \to +\infty$, such that $(\alpha', \beta') = g(\alpha, \beta)$.

Following Donaldson, we break the proof of Theorem 1 into two stages, summarized by the two propositions below.

Proposition 7. *The equivalence classes of complex trajectories associated with the homomorphisms* ρ_- *and* ρ_+ *are parametrized by* $\mathcal{N}(\rho_-) \cap S(\rho_+)$.

Proposition 8. (a) *For every complex trajectory* (α, β) *there is an equivalent trajectory* $(\alpha', \beta') = g(\alpha, \beta)$ *which satisfies the real equation* $\hat{F}(\alpha', \beta') = 0$ (*cf.* (10)(a)).

(b) *If* (α', β') *and* (α'', β'') *are equivalent complex trajectories, both satisfying the real equation* (10)(a), *then* $(\alpha'', \beta'') = g(\alpha', \beta')$ *for some* $g: \mathbf{R} \to G$ (*values in the compact group*) *with* $g(t) \to 1$ *as* $t \to +\infty$.

Let us spell out how Theorem 1 follows from these propositions. If $A(t)$ is a solution of the gradient-flow equations (1) satisfying the boundary conditions (2), then we obtain a complex trajectory (α, β) by the formulas (9), setting $A_0 = 0$. (The decay condition (iv) in Definition 5 is an easy consequence of the equations (1).) Thus we obtain a map from $M(\rho_-, \rho_+)$ to the space of equivalence classes of complex trajectories. Proposition 8(b) tells us that this map is injective. Now in each equivalence class there is a complex trajectory (α', β') satisfying the real equation (Proposition 8(b)), and on decomposing α' and β' into skew and self-adjoint parts according to (9), we obtain a solution (A_0, A_1, A_2, A_3) of the extended equations (7). Moreover A_0 decays exponentially, so there is a real gauge transformation $g: \mathbf{R} \to G$, with $g(t) \to 1$ as $t \to 1$, such that

$$\mathrm{Ad}(g)(A_0) - \frac{dg}{dt} g^{-1} = 0.$$

Thus by the transformation (8) we obtain a solution of the original equations, thus showing that the map from $M(\rho_-, \rho_+)$ to the complex trajectories is also surjective.

The two propositions are proved in §5 and §6 respectively.

5. Classification of complex trajectories

The essential point is that the complex equation (10)(b) is locally trivial: on any interval, we can always find a complex gauge transformation g which transforms a given solution (α, β) to a solution $(\alpha', \beta') = g(\alpha, \beta)$ with $\alpha' \equiv 0$ and $\beta'(t) = e^{-2t}\beta_0$, where $\beta_0 \in \mathfrak{g}^c$ is a constant. The only local invariant of a solution is therefore the conjugacy class of β_0. As a simple consequence, we have:

Lemma 9. *If* (α, β) *and* (α', β') *are complex trajectories which are equal outside some compact set* $K \subset \mathbf{R}$, *then they are equivalent in the sense of Definition* 6.

The next two lemmas exploit the boundary conditions at $-\infty$ and $+\infty$ respectively.

Lemma 10. *Let* (α, β) *be a solution of the complex equation satisfying the boundary conditions of Definition* 5 *at* $t = -\infty$. *Then there is a gauge transformation* $g_- : \mathbf{R} \to G^c$, *with* $g_-(t)$ *approaching a constant as* $t \to -\infty$, *such that* $(\alpha', \beta') = g_-(\alpha, \beta)$ *is the constant solution*:

$$2\alpha' = H_-, \qquad \beta' = Y_-.$$

Proof. There is no loss of generality in assuming that $g = 1$ in the boundary conditioin of Definition 5(iii). Thus 2α approaches H_- rapidly, and since the complex equation is trivial we can find a g_0, with $g_0(t) \to 1$ as $t \to -\infty$, such that

$$H_- = \mathrm{Ad}(g_0)(2\alpha) - \frac{dg_0}{dt} g_0^{-1}.$$

So we obtain a transformed solution $(\alpha'', \beta'') = g_0(\alpha, \beta)$ with $2\alpha'' \equiv H_-$. Using the complex equation, we find the most general possibility for β'':

$$\beta''(t) = \mathrm{Ad}(\exp(-(2 + H_-)t))(\omega),$$

for some $\omega \in \mathfrak{g}^c$. Via the adjoint representations of the elements H_-, X_-, Y_-, the vector space $\mathfrak{g}^c$ becomes a representation space for $\mathfrak{sl}(2, \mathbf{C})$. There is therefore a linear decomposition

$$\mathfrak{g}^c = \bigoplus_{i \in \mathbf{Z}} \mathfrak{g}^c(i),$$

where $\mathfrak{g}^c(i)$ denotes the eigenspace of $\mathrm{ad}(H_-)$ with eigenvalue i. Because $\beta'' \to Y$ as $t \to -\infty$, it is necessary that $\omega = Y_- + \delta$ for some

$$\delta \in \bigoplus_{i < -2} \mathfrak{g}^c(i).$$

The formula for β'' now becomes

$$\beta''(t) = Y_- + \mathrm{Ad}(\exp(-(2 + H_-)t))(\delta).$$

The gauge transformation g_0 was not uniquely determined: we still have the freedom to alter β'' by any gauge transformation g_1 which preserves the condition $2\alpha'' = H_-$ and approaches 1 at $t = -\infty$. The general solution of these constraints is

$$g_1(t) = \exp(-H_- t)\exp(\gamma)\exp(H_- t)$$

with $\gamma \in \bigoplus_{i<0} \mathfrak{g}^c(i)$. The action of g_1 on β'' has the effect of replacing

δ in the formula above by $\mathrm{Ad}(\exp(\gamma))(Y_- + \delta) - Y_-$. It therefore remains only to prove:

(11)
$$\text{for each } \delta \in \bigoplus_{i<-2} \mathfrak{g}^c(i), \text{ there exists } \gamma \in \bigoplus_{i<0} \mathfrak{g}^c(i) \text{ such that } \mathrm{Ad}(\exp(\gamma))(Y_- + \delta) - Y_- = 0.$$

This can be deduced from the implicit function theorem. However, since we shall prove a very similar statement (12) in the next lemma, we omit further details.

Lemma 11. *Let (α, β) be a solution of the complex equation satisfying the boundary conditions of Definition 5 at $t = +\infty$. Then there is a unique gauge transformation $g_+ : \mathbf{R} \to G^c$, with $g_+(t) \to 1$ as $t \to +\infty$, such that the transformed solution $(\alpha', \beta') = g_+(\alpha, \beta)$ satisfies*

$$2\alpha' = \text{const} = H_+, \qquad \beta'(0) \in S(\rho_+).$$

Proof. The first steps are just the same as the proof of Lemma 10. We find a gauge transformation g_0, approaching 1 at $t = +\infty$, such that the transformed solution (α'', β'') satisfies

$$2\alpha'' = H_+, \qquad \beta''(t) = Y_+ + \mathrm{Ad}(\exp(-(2 + H_+)t))(\varepsilon),$$

where

$$\varepsilon \in \bigoplus_{i>-2} \mathfrak{g}^c(i).$$

Here $\mathfrak{g}^c(i)$ now denotes the ith eigenspace of $\mathrm{ad}(H_+)$. We still have the freedom to alter β'' by a gauge transformation of the form

$$g_1(t) = \exp(-H_+ t)\exp(\gamma)\exp(H_+ t),$$

where $\gamma \in \bigoplus_{i>0} \mathfrak{g}^c(i)$ (note the change of sign). Such a gauge transformation has the effect of replacing ε by $\mathrm{Ad}(\exp(\gamma))(Y_+ + \varepsilon) - Y_+$. In view of the definition of $S(\rho_+)$, it only remains to prove:

(12)
$$\text{for each } \varepsilon \in \bigoplus_{i>-2} \mathfrak{g}^c(i), \text{ there exists a unique } \gamma \in \bigoplus_{i>0} \mathfrak{g}^c(i) \text{ such that } \mathrm{Ad}(\exp(\gamma))(Y_+ + \varepsilon) - Y_+ \in z(X_+).$$

If we expand the left-hand side near $\gamma = \varepsilon = 0$, the terms linear in γ and ε are

$$f(\gamma, \varepsilon) = \varepsilon - [Y_+, \gamma].$$

Since $Y_+ \in \mathfrak{g}^c(-2)$, the adjoint action of Y_+ defines a linear map

$$\mathrm{ad}(Y_+): \bigoplus_{i>0} \mathfrak{g}^c(i) \to \bigoplus_{i>-2} \mathfrak{g}^c(i).$$

(Remember that the Jacobi identity implies $[\mathfrak{g}^c(i), \mathfrak{g}^c(j)] \subset \mathfrak{g}^c(i+j)$.) It follows easily from the representation theory of $\mathfrak{sl}(2, \mathbf{C})$ that this linear map is injective and that its image is a complement of $z(X_+)$. So for every ε there exists a unique γ with $f(\gamma, \varepsilon) \in z(X_+)$; or in other words, the linearization of problem (12) admits a unique solution. By the implicit function theorem, then, for any sufficiently small ε, there exists a unique small γ satisfying (12). To dispense with the word 'small' in this statement, we exploit the homogeneity of our problem: the adjoint action of the one-parameter subgroup $\exp(-(2 + H_+)t)$ has positive weights on the space $\bigoplus_{i>-2} \mathfrak{g}^c(i)$, and condition (12) is preserved by the action

$$\varepsilon \mapsto \mathrm{Ad}(\exp(-(2+H_+)t(\varepsilon)), \qquad \gamma \mapsto \mathrm{Ad}(\exp(-H_+ t))(\gamma);$$

so the existence and uniqueness for large ε and γ follow from the result for small ε and γ. This completes the proof of Lemma 11.

Combining Lemmas 10 and 11, we see that every complex trajectory is equivalent, in the sense of Definition 6, to a trajectory (α, β) satisfying the conditions:

$$(13) \qquad \left.\begin{aligned} \alpha(t) &= \tfrac{1}{2}H_-, \\ \beta(t) &= Y_-, \end{aligned}\right\} t \in (-\infty, 0], \qquad \left.\begin{aligned} \alpha(t) &= \tfrac{1}{2}H_+, \\ \beta(t) &= Y_+ + \mathrm{Ad}(\exp(-(2+H_+)t))(\varepsilon), \end{aligned}\right\} t \in [1, \infty).$$

Furthermore we can arrange that $Y_+ + \varepsilon \in S(\rho_+)$, and ε is then uniquely determined. Since (α, β) is locally equivalent to the constant solution $(\frac{1}{2}H_-, Y_-)$, the element $Y_+ + \varepsilon$ must be conjugate to Y_- in $\mathfrak{g}^c$; in other words, $Y_+ + \varepsilon \in \mathscr{N}(\rho_-)$. Conversely, given any $Y_+ + \varepsilon$ in $S(\rho_+) \cap \mathscr{N}(\rho_-)$ we can construct a complex trajectory (α, β) satisfying conditions (13). Together with Lemma 9, these observations complete the proof of Proposition 7.

6. The real equation

In this section we shall prove Proposition 8 by adapting the analysis from §2 of [2]. Because of the equivalence between Nahm's equations and the equations (1), we can omit many calculations. We write $(\alpha', \beta') = g(\alpha, \beta)$ and regard the real equation $\hat{F}(\alpha', \beta') = 0$ as an equation for g. As such, it is invariant under the group of real gauge transformations: it depends only on the projection of g as a path in $\mathscr{H} = G^c/G$. Using the 'polar decomposition' in G^c, we regard $\mathscr{H}$ as the space of self-adjoint elements of G^c which are positive definite in the adjoint representation. For each g we write

$$h = h(g) = g^* g$$

and regard this as a path in $\mathscr{H}$.

The first lemma is a local existence result, Proposition (2.8) from [2]:

Lemma 12. *If α and β satisfy the complex equation on an interval $[-N, N]$, then for any h_- and h_+ in $\mathscr{H}$ there exists a continuous $g: [-N, N] \to G^c$ with $h = h(g) = h_-, h_+$ respectively at $-N, N$ and such that $(\alpha', \beta') = g(\alpha, \beta)$ satisfies the real equation $\hat{F}(\alpha', \beta') = 0$ in $[-N, N]$.*

Next we adapt the differential inequality, Lemma (2.10) from [2]. For $h \in \mathscr{H}$, define

$$\Psi(h) = \log \max(\lambda_i) \in \mathbf{R},$$

where $\lambda_1, \cdots, \lambda_k$ are the eigenvalues of $\mathrm{Ad}(h)$. Since $\det(\mathrm{Ad}(h)) = 1$, $\Psi(h)$ is zero if and only if $h = 1$. Moreover, Ψ is positive and proper, and for h close to 1 there is an inequality

$$|h - 1| \leq K\Psi(h).$$

Lemma 13. *If $(\alpha', \beta') = g(\alpha, \beta)$ over some interval in $\mathbf{R}$, then, with $h = g^* g$,*

$$\frac{d^2}{dt^2}\Psi(h) + 2\frac{d}{dt}\Psi(h) \geq -2(|\hat{F}(\alpha, \beta)| + |\hat{F}(\alpha', \beta')|)$$

in the weak sense.

Proof. See [2, Lemma (2.10)]. The extra term in our inequality is easily accounted for. The norm of $\hat{F}$ on the right-hand side should be defined using the Killing form.

We can immediately deduce the uniqueness result, part (b) of Proposition 8. For if (α', β') and (α'', β'') both satisfy the real equation, and $(\alpha'', \beta'') = g(\alpha', \beta')$ for some bounded complex gauge transformation g, with $g(t) \to 1$ as $t \to +\infty$, then we have

$$\ddot{\Psi} + 2\dot{\Psi} \geq 0,$$

where $\Psi = \Psi(h(g))$. Furthermore Ψ is bounded and approaches 0 at $t = +\infty$. This implies $\Psi \equiv 0$ (an easy exercise), so $h \equiv 1$, and g actually takes values in the compact group.

We turn to the existence result, part (a) of Proposition 8. Let (α, β) be a complex trajectory associated with homomorphisms ρ and ρ_+. By the results of §5, we may assume without loss of generality that (α, β) satisfies conditions (13). For such an (α, β) we make the following elementary observation.

Lemma 14. *If* (α, β) *are as in* (13) *and* $\varepsilon \in z(X_+)$,

$$\begin{cases} \hat{F}(\alpha, \beta) = 0 & \text{on } (-\infty, 0], \\ |\hat{F}(\alpha, \beta)| \leq Ce^{-4t} & \text{on } [0, \infty). \end{cases}$$

Proof. The first statement follows from the fact that ρ_- is a representation of $\mathfrak{su}(2)$. For the second statement, we compute, for $t > 1$,

$$\hat{F}(\alpha, \beta) = 2([Y_+, \varepsilon(t)^*] + [\varepsilon(t), Y_+^*] + [\varepsilon(t), \varepsilon(t)^*]),$$

where $\varepsilon(t) = \mathrm{Ad}(\exp(-(2 + H_+)t))(\varepsilon)$. Now ε lies in $z(X_+)$ and $X_+ = Y_+^*$. So only the last term is nonzero, and

$$|\hat{F}| = 2|[\varepsilon(t), \varepsilon(t)^*]| \leq Ce^{-4t}.$$

For each positive N we can find a complex gauge transformation g_N: $[-N, N] \to G^c$ such that $g_N(\alpha, \beta)$ satisfies the real equation. By Lemma 12 we can arrange further that $h_N = g_N^* g_N$ satisfies the Dirichlet boundary condition $h_N(\pm N) = 1$. Using the elementary estimate above and the differential inequality, Lemma 13, we shall show that the h_N have a smooth limit as $N \to \infty$.

Lemma 15. *If* C *is the constant from Lemma* 14 *and* $\psi: \mathbf{R} \to \mathbf{R}^+$ *is the* C^1 *function*

$$\psi(t) = \begin{cases} C/4, & t \leq 0, \\ Ce^{-2t}/2 - Ce^{-4t}/4, & t \geq 0, \end{cases}$$

then for all N *we have* $\Psi(h_N) < \psi$ *on* $[-N, N]$.

Proof. The function ψ is so constructed that

$$\ddot{\psi} + 2\dot{\psi} = \begin{cases} 0, & t < 0, \\ -2Ce^{-4t}, & t > 0. \end{cases}$$

So by Lemmas 13 and 14 we have $\ddot{\Psi} + 2\dot{\Psi} \geq \ddot{\psi} + 2\dot{\psi}$ on $[-N, N]$, where $\Psi = \Psi(h_N)$. The maximum principle implies that $\Psi - \psi$ takes its maximum value at one of the endpoints; but $\Psi - \psi$ is negative at $\pm N$, so $\Psi < \psi$ everywhere.

This lemma provides a uniform bound on h_N. Once one has this, another application of Lemma 13 shows that the h_N converge uniformly on compact subsets. Finally, using the fact that h_N satisfies an elliptic equation, one deduces

Corollary 16. (i) *The h_N converge in the C^∞ topology, on compact subsets, to a smooth path $h: \mathbf{R} \to \mathscr{H}$.*

(ii) *The path h is bounded, and for large t,*

$$|h(t) - 1| < \text{const}\, e^{-2t}.$$

(iii) *If $g = h^{1/2}$ and $(\alpha', \beta') = g(\alpha, \beta)$, then (α', β') satisfies the real equation $\hat{F}(\alpha', \beta') = 0$.*

We would be finished if we could show that (α', β') satisfied the boundary conditions of Definition 5, and so was a complex trajectory according to the definition. It may be necessary, however, to apply another (real) gauge transformation to (α', β') before this is the case. First, another estimate from [2]:

Lemma 17. *The derivative dh/dt is bounded, and for large t,*

$$|dh/dt| < \text{const}\, e^{-2t}.$$

Proof. Omitted, but see Lemma (2.20) from [2]; h satisfies an elliptic equation, so on any interval we can estimate dh/dt in terms of h and $\hat{F}(\alpha, \beta)$.

From this last lemma we deduce that $(\alpha', \beta') - (\alpha, \beta)$ decays exponentially as $t \to +\infty$; so the boundary condition of Definition 5(ii) holds. To deal with the boundary condition at $t = -\infty$, decompose (α', β') into self-adjoint and skew-adjoint parts according to (9), so as to obtain a solution (A_0, A_1, A_2, A_3) of the equations (7). Now make a real gauge transformation so as to make $A_0 \equiv 0$, thus obtaining a solution (A_1', A_2', A_3') of the gradient-flow equations. By Lemma 17, this is a bounded trajectory, and must therefore approach a critical point. This means that the boundary condition of Definition 5(iii) is satisfied, but perhaps with the wrong representation ρ_-. This last possibility need not worry us however: by Proposition A3 and Lemma 10, the conjugacy class of the representation ρ_- occurring as the limit point is uniquely determined by the orbit in which $\beta'(t)$ lies; and this is the same orbit which contained the original $\beta(t)$.

Appendix

As before, $\mathfrak{g}$ will denote the Lie algebra of a compact, semisimple group G, and $\mathfrak{g}^c$ will denote the Lie algebra of the complex form G^c. The first part of the following proposition is the Jacobson-Morosov Theorem; the second part is due to Kostant. Proofs of both parts are in [6].

Proposition A1. (i) *For every nilpotent element* $Y \in \mathfrak{g}^c$, *there exists a Lie algebra homomorphism* $\rho\colon \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{g}^c$ *such that* $Y = \rho\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

(ii) *If* $\rho, \rho'\colon \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{g}^c$ *are two homomorphisms and*

$$\rho\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \rho'\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then ρ *is conjugate to* ρ': *that is, there exists* $g \in G^c$ *such that* $\rho' = \mathrm{Ad}(g) \circ \rho$.

The second part of Proposition A1 has a companion, also proved in [6], which we used in §3.

Lemma A2. *If* $\rho, \rho'\colon \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{g}^c$ *are two homomorphisms and*

$$\rho\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \rho'\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then ρ *is conjugate to* ρ'.

What we have used on a few occasions is a variant of Proposition A1, in which the compact forms replace the complex ones. To deduce Proposition A3 from the previous version, it is only necessary to note that, for any compact group K (such as SU(2)), the classification of homomorphisms $K \to G$ up to conjugacy in G in the same as the classification of homomorphisms $K \to G^c$ up to conjugacy in G^c. (The author is grateful to A. Borel for pointing out how this statement may be proved.) We use the notation of §2:

Proposition A3. *The assignment of* $\mathcal{N}(\rho)$ *to* ρ *sets up a one-to-one correspondence between the conjugacy classes of homomorphisms* $\rho\colon \mathfrak{su}(2) \to \mathfrak{g}$ *and the* G^c*-orbits of nilpotent elements in* $\mathfrak{g}^c$.

Example. $G = \mathrm{SU}(n)$. Since $\mathfrak{su}(2)$ has one irreducible representation in each dimension, the homomorphisms $\rho\colon \mathfrak{su}(2) \to \mathfrak{su}(n)$ are classified, up to conjugacy, by the partitions of n. On the other hand, the partitions of n also classify the similarity classes of nilpotent n-by-n matrices, via their Jordan canonical forms.

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